MARTINGALES AND BROWNIAN MOTION

Exercise sheet 2: Discrete (sub-/super-)martingales

1. Theory

Within this section $(X_n)_n$ is a martingale with respect to a given filtration $(\mathcal{F}_n)_n$.

Exercise 1. *Show that for all $m \geq n$, we have that $\mathbb{E}[X_m|\mathcal{F}_n] = X_n$. Deduce that $(\mathbb{E}[X_n])_n$ is a constant sequence.*

Exercise 2. *Show that $(X_n)_n$ is also a martingale with respect to its natural filtration $(\mathcal{G}_n)_n$, i.e. with $\mathcal{G}_n := \sigma(X_1, \dots, X_n)$ for any n.*

Exercise 3. *Suppose that $(Y_n)_n$ is a sequence of i.i.d. random variables satisfying $\mathbb{E}[Y_i] = 0$ and $\mathbb{E}[Y_i^2] = \sigma^2$, where $\sigma > 0$ is a constant that does not depend on n. Set $S_n := \sum_{i=1}^n Y_i$. Show that $S_n^2 - \sigma^2 n$ is a martingale with respect to its natural filtration.*

Exercise 4. *Let $(Y_n)_n$ and $(Z_n)_n$ be a sub- and supermartingale, respectively. Prove the following statements.*

(1) If the sequence $(\mathbb{E}[Y_n])_n$ is constant, then $(Y_n)_n$ is a martingale. Conclude the same for $(Z_n)_n$.

(2) For any $a \in \mathbb{R}$,

a) $(\max(a, Y_n))_n$ is a submartingale;

b) $(\min(a, Z_n))_n$ is a supermartingale.

*Now additionally assume that the law of Y_n is the same for all n. Convince yourself from the above facts that then $(Y_n)_n$ is a martingale, which is called **equidistributed**.*

(3) Deduce that for all $a \in \mathbb{R}$ and $p > n \in \mathbb{N}$,

$$\{Y_n \geq a\} \subset \{Y_p \geq a\} \text{ up to a zero set.}$$

(4) Conclude that

$$Y_1 = Y_2 = Y_3 = \dots \quad \text{almost surely.}$$

2. Applications

Exercise 5. *Let $(X_n)_n$ be a sequence of i.i.d. integrable random variables with $\mathbb{E}[X_i] = 0$, and fix $N \in \mathbb{N}$. Define $M_n := \frac{1}{N-n} \sum_{i=1}^{N-n} X_i$. Show that $(M_n)_{n \leq N}$ is a martingale with respect to its natural filtration[1].*

Exercise 6. *Suppose that there are red and green balls in the urn. Repetitively we randomly draw a ball from the urn, and place it back together with a new ball of the same color. Assume*

[1] Convince yourself that it is equivalent to the natural filtration associated to $\left(\sum_{i=1}^{N-n} X_i\right)_{n \leq N}$.

that we start with the configuration of one green and one red ball. Let R_n be the number of red balls in the urn after n steps. Show that

$$\left(S_n := \frac{R_n}{n+2} \right)_n ,$$

is a martingale with respect to the natural filtration associated to $(R_n)_n$.

Exercise 7. *Let Z be a random variable uniformly distributed on $[0,1]$. For $n \geq 1$ and $k \in \{0, \ldots, 2^n\}$, we define $X_n := k2^{-n}$ if $k2^{-n} \leq Z < (k+1)2^{-n}$. Let $f : [0,1] \to \mathbb{R}$ a bounded function and $Y_n = 2^n(f(X_n + 2^{-n}) - f(X_n))$. Show that (Y_n) is a martingale with respect to the natural filtration associated to (X_n).*